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Multiscaling of the graph length in Coupled Map Lattices

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Abstract. We consider generalized scaling exponents for the graph of a scalar function to allow for multiscaling of the graph length. These generalized exponents are related to the moments of the distribution of roughness over the support of the graph. We report numerical computations of these exponents for the graphs of Coupled Map Lattices exhibiting phase transitions between laminar and turbulent behaviour. We observe multiscaling of the graph in the transition region. Furthermore, the generalized-dimensions $D(q)$ are computed for a conserved measure of graph roughness.

1. Introduction

Graphs of non-analytic scalar functions exhibiting wrinkling on many scales are encountered in many contexts, notably in turbulence. It has been experimentally observed (Sreenivasan, 1991) and theoretically argued (Constantin and Procaccia 1992, Procaccia and Constantin 1993, Constantin and Procaccia 1993) that the vorticity field, or such passive scalars as the concentration of a pollutant, or the temperature, exhibit fractal graphs between some lower cut-off length provided by the dissipation mechanism (say diffusion) present in the system and some typical length scale L over which the maximum variations in the value of the function are encountered. The graph is self affine; extending the domain to many times the scale L yields necessarily a flat surface at large scales.

The degree of roughness of the graph need not be uniformly distributed (figure 1). In particular, a properly defined capacity-dimension could be smaller than the dimension of the support (Constantin and Procaccia 1992), paving the way for the anomalous scaling of the structure functions and of the fields related to the derivatives of the graph under consideration.

In this paper we would like to investigate whether multiscaling of the graph length can be observed in Coupled Map Lattices (Kaneko 1985). Coupled Map Lattices can be thought of as modelling dynamical systems described by differential equations (Keeler and Farmer 1986). They are capable of exhibiting phase transitions between 'turbulent' and 'laminar' behaviour (Chaté and Manneville 1988, Houlrik *et al* 1990) and spatio-temporal intermittency along the phase boundary (Hüner and Erzan 1994).

The paper is organized as follows. In section 2 we recall the definition of the graph dimension and discuss how to compute it for a function defined over a discrete lattice. In section 3 we define scaling exponents for the qth moments of the graph length, averaged over time and space, and find numerically that for Coupled Map Lattices undergoing dynamical

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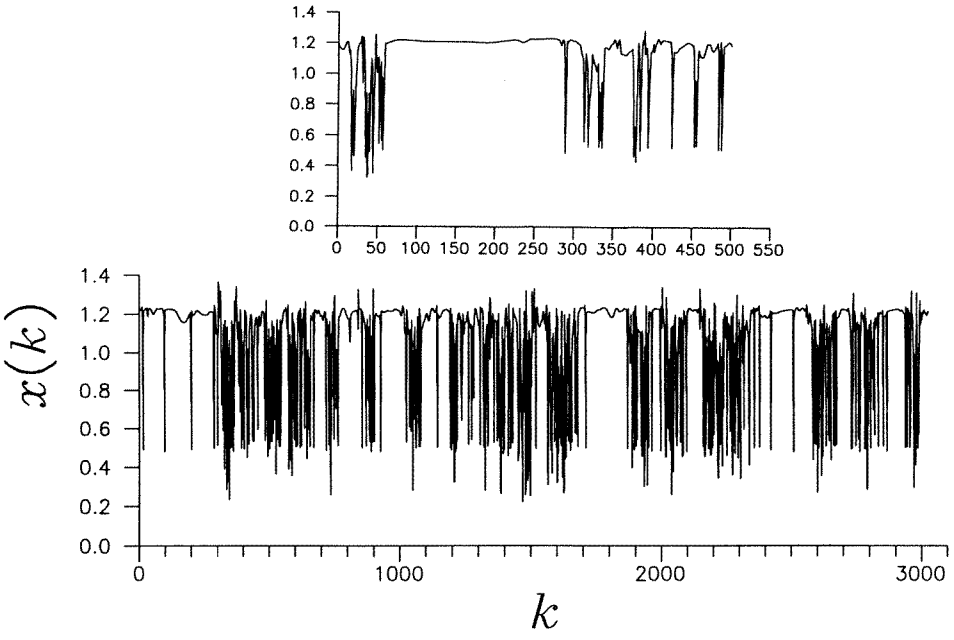

Figure 1. Graph of the coupled modified tent map, equation (18), for two different chains of length 3100 and 500, showing the self-similar, inhomogeneous distribution of the roughness. $\nu = 3$, $\epsilon = 0.3598$.

phase transitions, these can depend nontrivially on q. We then evoke the generalized dimensions $D(q)$ (Grassberger and Procaccia 1984, Hentschel and Procaccia 1983) to describe the distribution of the graph roughness.

2. Scaling of the graph length

After Constantin and Procaccia (1993) we consider the graph G of a scalar function $x(r)$, defined on a one-dimensional domain B of size R,

$$G(r) = (r, x(r)) \qquad r \in B \tag{1}$$

with a characteristic scale $L << R$ over which the largest variations in $x(r)$ of size ξ, are registered. We assume that the function $x(r)$ is bounded and continuous. The graph length (Falconer 1985) over a region of size L may then approximated to better and better accuracy by taking the finite differences $|x(r+\ell) - x(r)|$, with ℓ successively smaller, and forming the sums,

$$I(\ell, L) = \sum_{k=1}^{L/\ell} \Delta_k(\ell, L) \tag{2}$$

where

$$\Delta_k(\ell, L) = \sqrt{\ell^2 + \left(\frac{L}{\xi}\right)^2 (x(r_k + \ell) - x(r_k))^2} \tag{3}$$

with $r_k = (k-1)\ell$. The factor of (L/ξ) gives Δ the dimension of length, ξ is the absolute value of the maximum variation of $x(r)$ taken over the domain of size L.

In many applications in which one is interested, the graph may be smooth below some cut-off length ρ_0, or, as in the case we will consider below, the function $x(r)$ may only be defined over a discrete lattice with spacing which can be taken to be equal to ρ_0. We will then consider the graph to consist of the piecewise linear set of points obtained by connecting the values $x_k = x(k\rho_0)$ lying above the vertices of the lattice. Then, for $\ell = \rho_0$, $|x(r_k + \rho_0) - x(r_k)| \leqslant (\text{constant})\rho_0$ by definition. On the other hand, as ℓ becomes of the same order as L, $|x(r_k + \ell) - x(r_k)| \leqslant \xi(\frac{\ell}{L})$. For $\ell \geqslant L$, the graph length will no longer depend on ℓ. However, consider the possibility that for some intermediate range of scales $\rho_0 < \ell < L$, the finite difference in equation (3) scales like

$$|x(r_k + \ell) - x(r_k)| \leqslant \xi \left(\frac{\ell}{L}\right)^{2-s} \tag{4}$$

with $1 \leqslant s < 2$ (Falconer 1985). We see that we then have to distinguish between two cases. In equation (3) we can pull out a factor of ℓ outside the radical. Then, for $s = 1$, the expression inside the radical does not depend on ℓ any more and

$$\Delta(\ell, L) \sim \ell \qquad s = 1. \tag{5}$$

Correspondingly, the graph length given by the sum in equation (3) simply grows with L for $s = 1$. However, for $s > 1$ one gets from equation (4),

$$\Delta(\ell, L) \leqslant \ell \sqrt{1 + \left(\frac{L}{\ell}\right)^2 \left(\frac{\ell}{L}\right)^{2(2-s)}} \tag{6}$$

so that as (ℓ/L) gets progressively smaller than one, the second term inside the radical becomes larger and larger, and dominates the first. For fixed L,

$$\Delta(\ell, L) \leqslant \ell \left(\frac{\ell}{L}\right)^{(1-s)} \qquad 1 < s < 2 \tag{7}$$

and there is a correction to naive scaling of the graph length. In this case, $I(\ell, L)$ depends on ℓ and grows faster than linearly with L. In summary, from (2) and (7) one obtains,

$$I(\ell, L) = \begin{cases} \sim L & s = 1 \\ \leqslant L \left(\dfrac{L}{\ell}\right)^{(s-1)} = \ell \left(\dfrac{L}{\ell}\right)^{s} & 1 < s < 2. \end{cases} \tag{8}$$

Let us define the scaling exponent β by

$$I(\ell, L) \sim \ell^{1-\beta} L^{\beta} \tag{9}$$

for $\rho_0, \ell < L$; so that if one measures the graph length with sticks of size ℓ and of size ℓ/b, one will find $I(\ell/b, L)/I(\ell, L) \sim b^{\beta-1}$. We see that $\beta = 1$ for $s = 1$ and $\beta \leqslant s$ for $1 < s < 2$.

For the range of length scales over which it is well defined, the scaling exponent β can be related to the box counting-dimension of the graph, as defined by Falconer (1985) and by Procaccia and Constantin (1993) (also see Constantin and Procaccia 1993).

For a given set of linear size L, the box counting-dimension D is commonly defined via the scaling relation

$$N(\ell, L) \sim \left(\frac{L}{\ell}\right)^{D} \qquad \ell \to 0 \tag{10}$$

where $N(\ell, L)$ is the number of boxes of size ℓ needed to cover this set. Although, in many applications, as in our case, ℓ is bounded away from zero by a lower cut-off ρ_0, one speaks

of a self-similarity or box counting-dimension D in some finite scaling region $\rho_0 < \ell < L$ where $N(\ell/L)$ obeys such a power law as in (10). In this sense, one may estimate the box counting-dimension of the graph within a (one-dimensional) region of size L, from the number of boxes of size ℓ, $\rho_0 < \ell < L$, needed to cover the graph.

The number of sub-intervals of size ℓ needed to cover the region of size L is $\left(\frac{L}{\ell}\right)$, namely the number of terms in the sum in equation (2). The number of boxes of size ℓ needed to cover the piece of the graph lying above an ℓ-sized sub-interval can be estimated by dividing $\Delta_k(\ell, L)$, the end-to end distance of the graph over the kth sub-interval, by ℓ, i.e. $N_k(\ell, L) \leqslant \Delta_k(\ell, L)/\ell + 1$. The total number of boxes, $N(\ell, L)$, needed to cover the whole graph over the interval L is then

$$N(\ell, L) = \sum_{k}^{L/\ell} N_k(\ell, L) \leqslant \sum_{k} \Delta_k(\ell, L) + L/\ell. \tag{11}$$

Now note that, by definition (equations (3) and (10)),

$$I(\ell, L) \geqslant \ell N(\ell, L) - L \sim \ell^{1-D} L^D \tag{12}$$

from which we infer $\beta \geqslant D$ by (9). If, moreover, $\Delta(\ell, L)$ obeys such bounds as in equation (7), then one has $s \geqslant \beta \geqslant D$.

The scaling exponent β exhibits all the right properties one expects of a dimension. We see that (for a scalar graph defined over a one-dimensional domain) $1 \leqslant \beta \leqslant 2$. For a smooth graph, ($s = 1$), $\beta = 1$ and the length of the curve is simply proportional to the size of the interval, while for $s > 1$, there is a correction to scaling such that the 'length' grows faster with the size of the domain than linearly. For $s = 2$, such that $x(r)$ does not depend on r at all but is random, the graph can fill a strip, and the apparent-dimension for length scales $\rho_0 < \ell < L$ is 2.

Note that in this problem, there are in fact two different physical length scales in question. The characteristic scale L over which the largest variations in x are registered is, in effect, a decorrelation length for $x(r)$. The second length scale is the lower cut-off ρ_0 below which the graph becomes smooth. On the other hand, one may consider the scaling properties of the graph either in terms of the size R of an arbitrary region over which the graph is defined, or the size of the yardstick ℓ, $\rho_0 < \ell < R$, in terms of which the graph length is measured.

For $R \leqslant L$, $\rho_0 < \ell < R$, we may replace L with R everywhere in equations (2)–(8), so that we recover (9) with

$$I(\ell, L) \sim \ell^{1-\beta} R^{\beta}.$$

For regions of size $R >> L$, on the other hand, the graph length must again be simply proportional to the size of the region. Thus, it is useful to define a crossover function $f(y)$ such that the graph length as a function of R goes like, $\sim Rf(L/R)$, with

$$f(y) \sim \begin{cases} 1 & y \ll 1 \\ y^{\beta} & y \gg 1. \end{cases} \tag{13}$$

For $y \gg 1$, we may identify R with ℓ, in which case we will again recover equation (9). For $y \ll 1$, the $\Delta(\ell, L)$ do not depend on R, so that the sum in equation (2) is indeed linearly proportional to R. If we assume that the scaling behaviour in all the different L intervals that fit into the R domain are the same, we find,

$$I_L(\ell, R) \sim \frac{R}{L} \ell^{1-\beta} L^{\beta} \sim R \left(\frac{L}{\ell}\right)^{\beta-1} \tag{14}$$

in agreement with (13).

In systems undergoing dynamical phase transitions, such as those modelled by Coupled Map Lattices (Chaté and Manneville 1988, Houlrik *et al* 1990), we know that there are large stretches over which the function of interest may be very smooth ('laminar' regions), interrupted by ('turbulent') bursts of all sizes, over which the graph is extremely wrinkled. The support of the 'turbulent' patches right at the transition point between completely laminar and fully turbulent behaviour is a fractal (Hüner and Erzan 1994) up to length scales comparable to L, the decorrelation length. Clearly one needs to define averaged quantities over the whole domain. In fact, the graphs in which one is interested are also functions of time and we would like scaling relations like equation (9) to be applicable to time averaged quantities.

We have computed the scaling exponent β for two different chains of diffusively coupled nonlinear maps of the form (Kaneko 1985)

$$x_i^{(n+1)} = (1-\epsilon)g(x_i^{(n)}) + \frac{\epsilon}{2}[g(x_{i+1}^{(n)}) + g(x_{i-1}^{(n)})] \tag{15}$$

where the superscripts indicate the time step and where the local nonlinear transformation rule g specifies the particular model. Computations were performed on chains of length 480–3360.

2.1. Model A; the modified tent map

$$g(u) = \begin{cases} \nu u & 0 \leqslant u \leqslant \frac{1}{2} \\ \nu(1-u) & \frac{1}{2} \leqslant u \leqslant 1 \\ u & u > 1 \end{cases} \tag{16}$$

for $\nu > 2$.

This single map has transient chaos for $0 < u < 1/\nu$, $1-1/\nu < u < 1$. From within the interval $[1/\nu, 1-1/\nu]$ the trajectory is kicked to the interval $[1, \nu/2]$ where it sticks. The coupled variables x_i have a nonzero chance to be re-injected into the 'turbulent' interval $(0, 1)$ once they are in the 'laminar' region $[1, \nu/2]$. The dynamics of this chain can be portrayed in an economical way by assigning a two-valued representation to the x_i according to whether they are in one or the other region (Chaté and Manneville 1988). There is a phase boundary in the ϵ, ν space, below which a finite fraction of the lattice points are in the 'turbulent' state, whereas above the phase boundary, eventually all points fall into the laminar region (Chaté and Manneville 1988, Houlrik *et al* 1990, Grassberger and Schreiber 1991). Along the phase boundary, one finds that the 'turbulent' set of points is a fractal (thus, the density of such points vanishes in the limit of an infinite chain), with a fractal dimension which depends upon ν and varying from 0.8 and 0.84. For chains of 512 sites. the box covering-dimension is found to be well defined over length scales ranging from 2 to 256 sites (Hüner and Erzan 1994).

For the Coupled Map Lattices with the modified tent map we found that along the 'turbulent' to 'laminar' phase boundary, the graph length scaled with a well-defined exponent for ℓ between 1–24 and β was found to take values between 1.43 and 1.58 ± 0.04, for different parameter values (see table 1). (L was determined by inspection to be ~ 144, in order to maximize the ℓ-interval over which the graph length scaled.)

2.2. Model B; the logistic map

$$g(u) = \lambda u(1-u) \tag{17}$$

Table 1. The scaling exponent $\beta(q)$ for the coupled modified tent map, for different parameter values in the intermittent region. The linear least squares fits are made on the interval $1 \leqslant \ell \leqslant 24$, and we report the chi-squared error estimates.

$\beta(q)$	$\varepsilon_c = 0.1$ $\nu = 2.539$	$\varepsilon_c = 0.3598$ $\nu = 3$	$\varepsilon_c = 0.7$ $\nu = 3.735$	$\varepsilon_c = 0.9083$ $\nu = 3$
$\beta(0)$	1.35 ± 0.01	1.33 ± 0.01	1.31 ± 0.01	1.43 ± 0.03
$\beta(1)$	1.53 ± 0.02	1.53 ± 0.02	1.43 ± 0.02	1.58 ± 0.04
$\beta(2)$	1.62 ± 0.02	1.64 ± 0.02	1.51 ± 0.02	1.65 ± 0.04
$\beta(3)$	1.65 ± 0.02	1.69 ± 0.02	1.55 ± 0.02	1.68 ± 0.04
$\beta(4)$	1.66 ± 0.01	1.71 ± 0.02	1.57 ± 0.02	1.70 ± 0.04
$\beta(5)$	1.66 ± 0.01	1.72 ± 0.02	1.57 ± 0.02	1.71 ± 0.04

with $0 < \lambda < 4$. Here we availed ourselves of a direct numerical computation (Çikci 1995) of the largest Lyapunov exponent for the Coupled Map Lattices with fixed $\epsilon = 0.5$ as a function of λ and considered values of λ such that the system was in the chaotic regime, with some points taken near periodic windows. In this case the decorrelation length L was very small, of the order of at most four lattice spacings, and therefore no region was found over which β was well defined.

3. Multiscaling of the graph length in Coupled Map Lattices

Consider now the graph over a domain of size $R \gg L$, which we break up into boxes of size L. In computing the scaling exponent β we have performed an average over these boxes, also averaged over many time steps separated by intervals of the order of the size of the system. Thus

$$\langle\langle I(\ell, L)\rangle\rangle = \left\langle \frac{1}{N_L} \sum_i^{N_L} I_i(\ell, L) \right\rangle \sim L^\beta \ell^{1-\beta} \tag{18}$$

where the single set of brackets denote a time average, and the double set time and space average, $N_L = R/L$.

This overall scaling exponent does not tell us, however, how uniformly or unevenly roughness is distributed over different boxes of size L covering the whole domain. We can get an idea about this by forming qth moments of $I(\ell, L)$ and defining the scaling exponents β_q via

$$\langle\langle I^q(\ell, L)\rangle\rangle = \left\langle \frac{1}{N_L} \sum_i^{N_L} I_i^q(\ell, L) \right\rangle \sim L^q \left(\frac{L}{\ell}\right)^{q(\beta_q - 1)} . \tag{19}$$

For q=1, we simply recover $\beta = \beta_1$; if the scaling behaviour of $I_i(\ell, L)$ is uniform over all the L boxes, one also has $\beta_q = \beta$ for all q. However, for a non-uniform distribution, in general we will have a nontrivial dependence on q. For larger q, those L boxes for which the graph length is greater (larger β) will have more weight in the sum (19); thus β_q will be a nondecreasing function of q. The limit $q \to 0$ yields,

$$\beta_0 \ln L + (1 - \beta_0) \ln \ell = \langle\langle \ln I(\ell, L)\rangle\rangle.$$

Thus one finds,

$$\beta_0 = \langle\langle \beta \rangle\rangle.$$

For $q > 1$, β_q is a generalized scaling exponent which is like an effective graph-dimension over those regions of the domain R with a higher and higher degree of roughness. For model A, for parameter values such that one is in the laminar region, $\beta_q = 1$ for all q. In the turbulent region the scaling relation breaks down. Along the phase boundary we obtain multiscaling behaviour for the graph length. The values found for β_q, together with the chi-squared errors, are reported in table 1, for the scaling region $1 \leqslant \ell \leqslant 24$. The slopes have been computed from a least squres fit to the averaged value of the qth moments (equation (19)), over 10 snapshots separated by 3000 timesteps. The chain length is 3360. The true errors are about twice as large due to fluctuations in time. Note that, the error bars cannot be reduced by taking longer chains, since the decorrelation length L does not increase with chain length. Nevertheless, it can be seen that there is a systematic increase in β_q with q, with nontrivial dependence on q.

For model B, we cannot speak of well-defined scaling exponents.

We now turn to the question of how the roughness is distributed over the chain and to this end avail ourselves of the generalized-dimensions $D(q)$ (Hentschel and Procaccia 1983, Halsey *et al* 1986). Consider forming the ratio,

$$\mu_k(\ell, L) = \frac{\Delta_k(\ell, L)}{I(\ell, L)}$$

with $\sum_k^{L/\ell} \mu_k(\ell, L) = 1$, and the partition function

$$\chi_L(q; \ell) = \left\langle\!\!\left\langle \sum_k^{L/\ell} \mu_k^q \right\rangle\!\!\right\rangle \sim \left(\frac{\ell}{L}\right)^{\tau(q)} \tag{20}$$

whereby we define the scaling exponent $\tau(q)$. The brackets again mean an average over time and the positions of the L-balls. If this partition function indeed scales with ℓ/L, for $\rho_0 < \ell < L$, one may compute the generalized-dimensions $D(q) = \tau(q)/(q-1)$. The limit $q \to 1$ gives the capacity-dimension

$$D(1) = \left\langle\!\!\left\langle \sum_k \mu_k \ln \mu_k \right\rangle\!\!\right\rangle / \ln(\ell/L). \tag{21}$$

It can be seen from figure 2 and table 2 that such scaling is indeed observed for model A (equations (15), (16)) along the phase boundary, within the scaling region $1 \leqslant \ell \leqslant 24$. The

Table 2. $D(q)$ of the roughness measure, for the coupled modified tent map, for different parameter values in the intermittent region. Chi-squared error estimates are given for the same ℓ interval as in table 1. The true errors are about twice as large due to fluctuations over time.

	$\varepsilon_c = .1$ $D(q)$ $\nu = 2.539$	$\varepsilon_c = 0.3598$ $\nu = 3$	$\varepsilon_c = 0.7$ $\nu = 3.735$	$\varepsilon_c = 0.9083$ $\nu = 3$
$D(-5)$	1.54 ± 0.01	1.55 ± 0.01	1.43 ± 0.02	1.54 ± 0.04
$D(-4)$	1.51 ± 0.01	1.51 ± 0.01	1.40 ± 0.01	1.50 ± 0.03
$D(-3)$	1.46 ± 0.01	1.46 ± 0.02	1.34 ± 0.01	1.44 ± 0.03
$D(-2)$	1.36 ± 0.01	1.36 ± 0.02	1.26 ± 0.01	1.35 ± 0.03
$D(-1)$	1.21 ± 0.01	1.21 ± 0.01	1.14 ± 0.01	1.21 ± 0.02
$D(0)$	1	1	1	1
$D(1)$	0.80 ± 0.01	0.83 ± 0.01	0.86 ± 0.005	0.82 ± 0.01
$D(2)$	0.60 ± 0.016	0.68 ± 0.01	0.72 ± 0.01	0.67 ± 0.01
$D(3)$	0.46 ± 0.01	0.54 ± 0.01	0.61 ± 0.01	0.57 ± 0.01
$D(4)$	0.39 ± 0.01	0.43 ± 0.01	0.52 ± 0.01	0.50 ± 0.02
$D(5)$	0.34 ± 0.01	0.35 ± 0.02	0.45 ± 0.01	0.45 ± 0.02

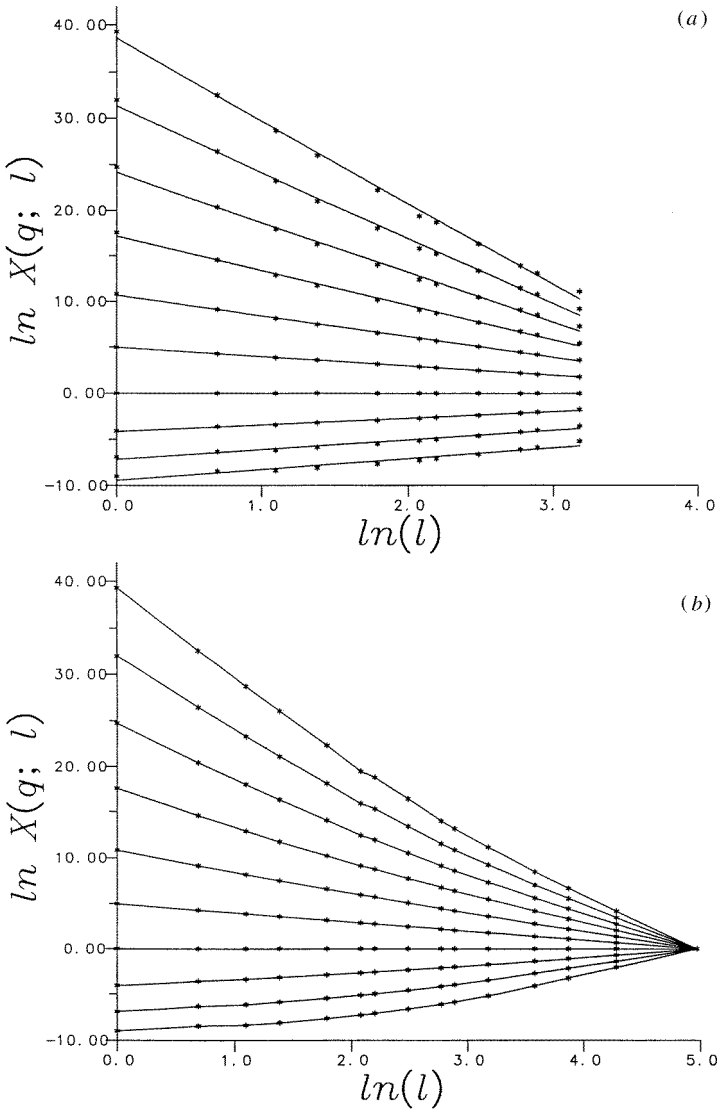

Figure 2. (*a*) Multiscaling of the partition function $\chi_L(q;\ell)$ for model A ($\nu = 3.0$, $\epsilon = 0.3598$). Here $L = 144$, $1 \leqslant \ell \leqslant 24$, and q varies form -5 to 4 from the top to bottom. The lines are best fits to the data points. The values of $D(q)$ are given in table 2. (*b*) Plot of the function $\ln \chi_L(q;\ell)$ $q = -5, -4, \ldots 4$, over the whole range $1 \leqslant \ell \leqslant L$, with a clear crossover behaviour around $\ell \sim 24$. The lines are a guide to the eye; see text.

$D(q)$ decrease monotonically with q; $D(0) = 1$ as it should, since the graph is defined over a one-dimensional chain. The capacity-dimension, (Farmer *et al* 1983) $D(1)$ ranges between 0.80 and 0.86 ($\pm$ .01) over the phase boundary, in agreement with the values found from numerical simulations for the fractal-dimension of the 'turbulent' set (Hüner and Erzan 1994). This indicates that the roughness indeed is indeed confined to the 'turbulent' points, a set of smaller-dimension than the whole chain. This is due to the very pronounced laminar stretches produced by the fixed line $[1, \nu/2]$ in the definition of $g(u)$ (equation (16)).

In all of the above, we found that the quality of the fits to single snapshots were quite good, but that the numbers tended to fluctuate somewhat in time. The fluctuations are once more of the order of the reported error bars.

An inspection of figure 2(*b*), where we display $\ln \chi_L(q;\ell)$ over the whole range $1 \leqslant \ell \leqslant L$ clearly shows a crossover behaviour around $\ln \ell \sim 3$. For $q = 0$, the plot is a straight line over the whole range, with slope equal to -1, giving $D(0) = 1$ as expected. The other straight line is for $q = 1$, since $\ln \chi_L(1;\ell) \equiv 0$ and $D(1)$ has to be computed separately from equation (21). For $24 < \ell$ the lines bend over, and one finds $\tau(q) = q - 1$, or $D(q) = 1$ for all q.

For the coupled logistic map, the dependence of $D(q)$ on q is much weaker; one finds $D(q)$ to be unity within the error bars.

We have considered the scaling behaviour of the qth moments of the graph length to account for the non-uniform wrinkling of graphs. We have shown that this generalization is meaningful by testing it on one-dimensional Coupled Map Lattices. We have found that these systems are capable of exhibiting multiscaling in the region where they are intermittent; in particular, the capacity dimension for the roughness is found to be less than the-dimension of the substrate.

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